Chapter 2 Algebra: The Basics

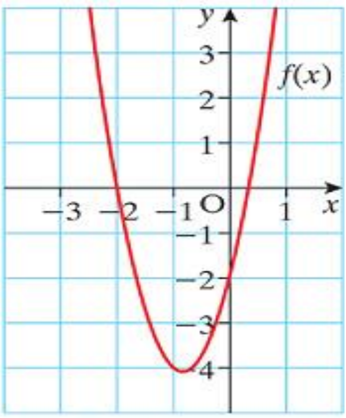
it's advisable to know/practice some of the examples in the book also.

Quadratic Equations:

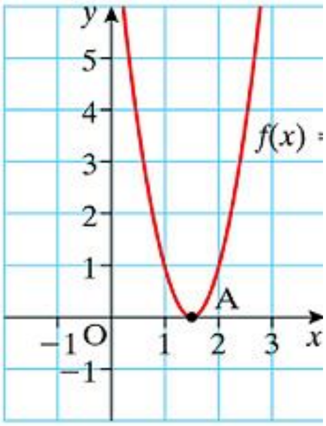
To factorise/solve these we use

- Guide Number
- Highest Common Factor
- Perfect Squares
- Quadratic (- b) Formula
- Graphical Methods

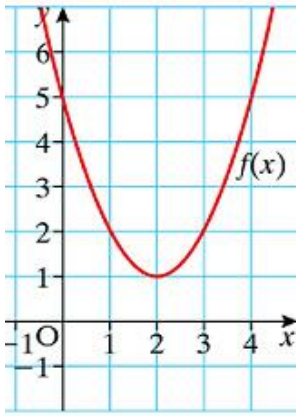
Types of Quadratic Roots: $(b^2 - 4ac)$ **is known as the DISCRIMINANT**



$b^2 - 4ac > 0$



$b^2 - 4ac = 0$



$b^2 - 4ac < 0$

In summary

1. If $(b^2 - 4ac) > 0 \rightarrow$ two different (distinct) real roots
2. If $(b^2 - 4ac) = 0 \rightarrow$ two equal real roots
3. If $(b^2 - 4ac) < 0 \rightarrow$ two imaginary roots
4. If $(b^2 - 4ac)$ is a perfect square $\rightarrow$ rational roots

NB: Roots are **real** if $(b^2 - 4ac) \geqslant 0$

Forming Quadratic Equations from their roots:

Given r_1, r_2 as the roots of an equation, then the equation is

$$x^2 - x(r_1 + r_2) + r_1 r_2 = 0,$$

i.e. $x^2 - x$ (sum of the roots) + product of the roots = 0.

Solving Quadratic and Linear Equations:

(note: this is also a method to find the point of intersection between a circle and a line)

Step 1: Rewrite the linear equation (or simpler of the two equations) as x =…. or y=…….

Step2: Sub this x = ….or y =… from step 1 into the OTHER EQUATION, simplify and solve it.

Step3: Sub the results from Step 2 into the first equation in step 1 to find the other letter

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Forming a quadratic given the roots:

$$x^2 - (sum\ of\ the\ roots)x + (product\ of\ the\ roots) = 0$$

Max & Min of a Quadratic Graph:

Completing the square:

The quadratic $\mathbf{x^2 - 6x + 11}$ can be rewritten as

$x^2 - 6x \mathbf{+ 9 - 9} + 11$ *(half the coefficient of the x value, square it, add and subtract it before the 11)*

$\underline{x^2 - 6x + 9} + 2 = 0$ (

$(x - 3)^2 + 2 = 0$

We can tell the following information from the equation written this way:

i) $a(x - p)^2 + q = 0$ $(x - 3)^2 + 2 = 0$

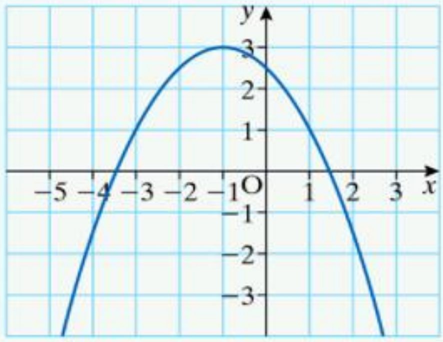
(p,q) is the minimum value of the graph which is (3,2) in this question

ii) $q - a(x - p)^2 = 0$ $2 - (x - 3)^2 = 0$

(p,q) is the maximum value of the graph, in this question it would be (3,2)

iii) By solving for x in $(x - 3)^2 + 2 = 0$ we can work out if there are real or Imaginary roots (see pg 63 in book 6)

(i) Write the equation of the graph provided in the form $y = q - a(x-p)^2$, where (p, q) is the maximum point of the curve and a is a constant.
(ii) By choosing any suitable point on the curve, find a.
(iii) Hence write the equation in the form $y = ax^2 + bx + c$.



(i) The maximum point $= (-1, 3) = (p, q)$.

$$y = q - a(x-p)^2$$
$$\therefore \quad y = 3 - a(x+1)^2$$

(ii) Selecting $(x, y) = (1, 1)$,

$$\Rightarrow 1 = 3 - a(1+1)^2$$
$$= 3 - 4a$$
$$\Rightarrow a = \tfrac{1}{2}$$

(iii) $\therefore \quad y = 3 - (\tfrac{1}{2})(x+1)^2$

$$\therefore \quad y = 3 - \frac{x^2}{2} - x - \frac{1}{2}$$
$$= -\frac{x^2}{2} - x + 2\tfrac{1}{2}$$

Surds:

1. Reducing surds to their lowest form

$\sqrt{5}, \sqrt{6}, \sqrt{7}$ cannot be simplified further,
but $\sqrt{8} = \sqrt{4 \times 2} = \sqrt{4} \times \sqrt{2} = 2\sqrt{2}$,
since 8 has a factor that is a perfect square.

$$\sqrt{ab} = \sqrt{a} \times \sqrt{b}$$

2. Simplifying surd quotients

$$\sqrt{\frac{50}{64}} = \frac{\sqrt{50}}{\sqrt{64}} = \frac{\sqrt{25} \times \sqrt{2}}{8} = \frac{5\sqrt{2}}{8}$$

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

3. Adding or subtracting surds

$$a\sqrt{b} \pm c\sqrt{b} = (a \pm c)\sqrt{b}$$

$2\sqrt{3} + 4\sqrt{3} = 6\sqrt{3}$.
$\sqrt{27} - \sqrt{12} = \sqrt{9 \times 3} - \sqrt{4 \times 3} = 3\sqrt{3} - 2\sqrt{3} = \sqrt{3}$.

4. Multiplying surds

$\sqrt{4} \times \sqrt{4} = (\sqrt{4})^2 = 4$.
$\sqrt{5} \times \sqrt{6} = \sqrt{30}$.
$(7 - \sqrt{2})(7 + \sqrt{2}) = 49 + 7\sqrt{2} - 7\sqrt{2} - 2 = 47$.

$$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$$

5. Dividing by surds

It is normal practice to not leave a surd (an irrational number) in the denominator of a quotient, hence the practice of "**rationalising the denominator**".

$$\frac{5}{\sqrt{3}} = \frac{5}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{5\sqrt{3}}{3}$$ (**Note:** Multiplying by $\frac{\sqrt{3}}{\sqrt{3}}$ is equivalent to multiplying by 1.)

$$\frac{1}{7-\sqrt{2}} = \frac{1}{7-\sqrt{2}} \cdot \frac{7+\sqrt{2}}{7+\sqrt{2}}$$
$$= \frac{7+\sqrt{2}}{7^2 - \sqrt{2}^2} = \frac{7+\sqrt{2}}{47}$$

To rationalise the denominator:
$$\frac{1}{a-\sqrt{b}} = \frac{1}{a-\sqrt{b}} \cdot \frac{a+\sqrt{b}}{a+\sqrt{b}}$$

Algebra Surd Equations:

Main rules/method:

- If there is **only one surd**, isolate it on one side and then square both sides and solve.
- If there are **two surds**, move one to each side of the equation. Square both sides and isolate any remaining surds. Square both sides again to remove any remaining surd.
- Solve the resulting equation.
- Check your answers.

Note: It is important to check <u>**ALL**</u> solutions in the ORIGINAL equation

<u>The Factor Theorem:</u>

The Factor Theorem:

If $f(k) = 0$, then $(x - k)$ is a factor.

Conversely, if $(x - k)$ is a factor, then $f(k) = 0$.

Also, if $(ax - k)$ is a factor, then $f\left(\frac{k}{a}\right) = 0$.

- A value is a root if we get 0 when we sub it into the equation
- Eg: If x = 2 is a root, then x – 2 is the factor
- Long division and solving will find other roots
- Use trial and error if no roots given to begin with

<u>Types of roots of Cubic Polynomials</u>

1. Three real roots

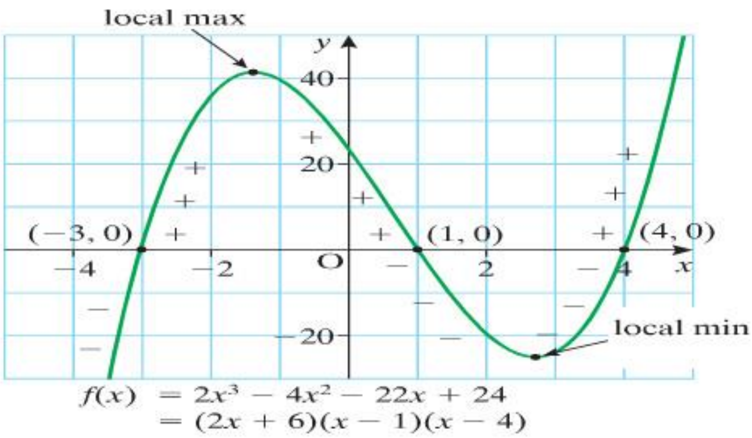
$f(x) = 2x^3 - 4x^2 - 22x + 24$

$f(x) = (x - 1)(2x + 6)(x - 4)$

This graph has three real roots, $-3, 1, 4$.

As the graph passes through a root, the value of the function changes from $(-)^{ve}$ to $(+)^{ve}$ or $(+)^{ve}$ to $(-)^{ve}$.

The graph has two turning points, a local maximum and a local minimum.



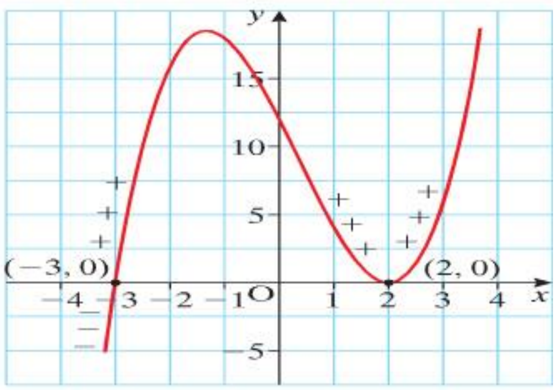
2. Three real roots, two of which repeat

$f(x) = x^3 - x^2 - 8x + 12$
$= (x + 3)(x - 2)^2$

This graph again has three real roots, $-3, 2, 2$, but one of the roots is repeated.

This graph only crosses the x-axis once because of the repeated root.

The graph has two turning points.



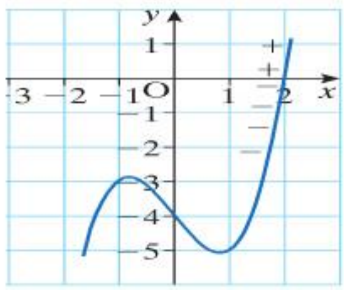
$f(x) = x^3 - x^2 - 8x + 12$
$= (x + 3)(x - 2)(x - 2)$
$= (x + 3)(x - 2)^2$

3. One real, two imaginary roots

$f(x) = x^3 - 2x - 4$
$= (x - 2)(x^2 + 2x + 2)$ ← using the quadratic formula
$= (x - 2)(x + 1 - \sqrt{-1})(x + 1 + \sqrt{-1})$

This polynomial has only one real root and two imaginary roots.

It crosses the x- axis once and has two turning points.



$f(x) = x^3 - 2x - 4$
$= (x - 2)(x^2 + 2x + 2)$

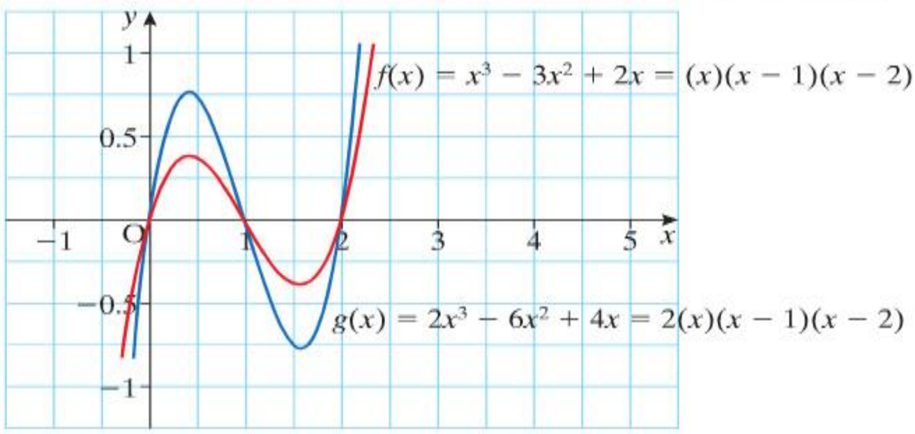
4. Comparing $f(x) = x^3 - 3x^2 + 2x$ and $g(x) = 2x^3 - 6x^2 + 4x = 2f(x)$

Both polynomials have the same roots, $x = 0, 1, 2$.

$\therefore$ the polynomials have common factors (x), $(x - 1)$ and $(x - 2)$.

But $g(x)$ has an integer factor 2 as well that multiplies each value of the curve, except where the value is zero at the roots.

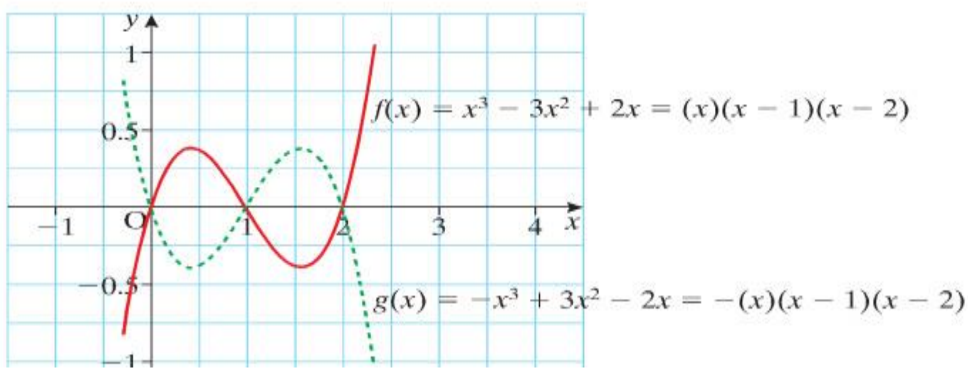
This integer factor acts as an amplification factor.



5. Comparing $f(x) = x^3 - 3x^2 + 2x$ and $g(x) = -x^3 + 3x^2 - 2x = -f(x)$

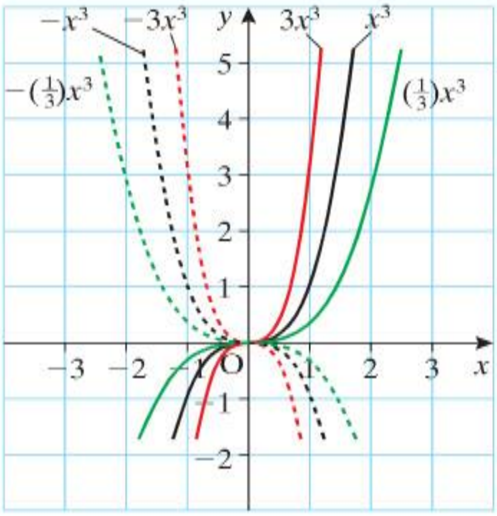
Again, both polynomials have the same roots and hence common factors.

The graphs are symmetrical across the x-axis.



6. The graphs of $f(x) = ax^3$

All the graphs pass through $(0, 0)$.
For $a > 0$, the graphs are all increasing, and as a increases, the graphs rise more steeply.
For $a < 0$, the graphs are decreasing.



Note:

(i) If $f(x) = 3x^3$ and $g(x) = -3x^3$,
$\Rightarrow f(x) = -g(x)$, i.e. $f(x)$ is the reflection of $g(x)$ in the x-axis.

(ii) $f(-x) = g(x)$, i.e. $f(x)$ and $g(x)$ reflect each other in the y-axis.

Note: There are no local maximum or minimum points as in the previous graphs.